A Unified Treatment for XXX-Heisenberg Model and Haldane-Shastry Model Using Shift Operators

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Abstract

A unified treatment is developed for the XXX-Heisenberg model and a long-ranged interaction model (the H_2 in Haldane-Shastry model) from the point of view of shift operators (or raising and lowering operators), based on which the energy spectra of the spin-chain models are determined. Some physical discussions are also made.

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I. Introduction and Motivation

Recently much attention has been paid to long-ranged interaction models [1]-[10]. This is because such models are believed to be relevant to the quantum Hall effect and high T_c superconductivity. The $S = 1/2$ Haldane-Shastry model (HSM) is one kind of typical and integrable long-ranged interaction models independently introduced by Haldane and Shastry [4][5]. The following mutually commuting Hamiltonians are found for the HSM in [11][12]:

$$H_2 = {\sum_{ij}}' \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right) (P_{ij} - 1),$$

$$H_3 = {\sum_{ijk}}' \left(\frac{z_i z_j z_k}{z_{ij} z_{jk} z_{ki}} \right) (P_{ijk} - 1),$$

$$H_4 = {\sum_{ijkl}}' \left(\frac{z_i z_j z_k z_l}{z_{ij} z_{jk} z_{kl} z_{li}} \right) (P_{ijkl} - 1) + H_4',$$

$$H_4' = -\frac{1}{3} H_2 - 2 {\sum_{ij}}' \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right)^2 (P_{ij} - 1), \tag{1}$$

where P_{ij} is an operator that exchanges the spins at sites i and j, P_{ijk} and P_{ijkl} are the cyclic permutation operators:

$$P_{ijk} = P_{ij} P_{jk} + P_{jk} P_{ki} + P_{ki} P_{ij},$$

$$P_{ijkl} = P_{ij} P_{jk} P_{kl} + \text{Cyclic}(i \to j \to k \to l).$$

The primed sum omits equal values of the summation variables and $z_{ij} \equiv z_i - z_j$, $z_j = \exp(i2\pi j/N)$. For spin $S = 1/2$, $P_{ij} = 1/2 + 2\vec{S}_i \cdot \vec{S}_j$. In this case, H_2, the first member of the Hamiltonian family of HSM reads ($H_{hs} \propto H_2$, and $N =$ even for simplicity)

$$H_{hs} = 4 \sum_{i<j}^{N} J_{ij} \left(\vec{S}_i \cdot \vec{S}_j - \frac{1}{4} \right), \quad J_{ij} = J_0 / \sin^2[(j-i)\pi/N], \tag{2}$$

whose ground-state energy and correlation functions have been obtained together with the thermodynamics [4]-[6]. The ground-state wave function is a spin singlet of the Jastrow-Gutzwiller form. The excitations are spin-1/2 spinons that form a gas of a semionic nature [4]. Furthermore, the H_{hs} is a natural extension of the XXX-Heisenberg model (XHM) [13]−[22] defined by

$$H_{xxx} = J \sum_{j=1}^{N} \left(\vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j+1} - \frac{1}{4} \right), \tag{3}$$

which is a nearest-neighbor interaction model exactly solved by the traditional Bethe ansatz [23].

It can be observed that, eigenfunctions of spin-1/2 chain models (such as H_{xxx} and H_{hs}) with $r(r = 0, 1, 2, \cdots, N)$ down-spins (i.e. with the eigenvalue $S_z = \frac{N}{2} - r$, where S_z is the z-component of the total spin) have always the same form, they are

$$| \psi_r >= \sum_{m_1 < m_2 < \cdots < m_r} a(m_1, m_2, \cdots, m_r) \phi(m_1, m_2, \cdots, m_r), \tag{4}$$

where $\phi(m_1, m_2, \cdots, m_r)$ represents a spin state with r down-spins on the m_j-th ($j = 1, 2, \cdots, r$) sites. In particular, $| \psi_0 >=|\uparrow\uparrow \cdots \uparrow>$ is the vacuum state with all spins up of the spin-chain models. $a(m_1, m_2, \cdots, m_r)$ are coefficients defined for the ordering $m_1 < m_2 < \cdots < m_r$ and depend only upon $m_1, m_2, \cdots, m_r$. The determination of $a(m_1, m_2, \cdots, m_r)$ will be depended on particular model, i.e., XHM or HSM. The fact that eigenfunctions of different spin-chain models share the same form prompts us to develop a unified treatment for these models.

Although the XHM and HSM have been studied for a long time, to our knowledge, there has not been a unified treatment yet. The purpose of this paper is to provide a unified treatment for the H_{xxx} and H_{hs} from the point of view of shift operators (or raising and lowering operators), based on which the energy spectra of the models are determined. The paper is organized as follows. In Sec. II, since the eigenfunctions of H_{xxx} and H_{hs} have the same form [see Eq. (4)], so we shall focus on Eq. (4) and determine the forms of shift operators from physical considerations, and provide clear physical pictures how the raising operator $Q_{r,r-1}^+$ (or lowering operator $Q_{r-1,r}^-$) transforms $| \psi_{r-1} >$ to $| \psi_r >$ (or $| \psi_r >$ to $| \psi_{r-1} >$). In Sec. III and VI, we shall derive explicit results of shift operators for H_{xxx} and H_{hs} through the following definition of shift operators in a commutator form:

$$([H, Q_{r,r-1}^+] = \omega_{r,r-1} Q_{r,r-1}^+) \ | \psi_{r-1} >, \ \ ([H, Q_{r-1,r}^-] = \omega_{r-1,r} Q_{r-1,r}^-) \ | \psi_r >, \tag{5}$$

where $\omega_{nm} = E_n - E_m$ is the energy interval, E_r is the energy corresponding to $| \psi_r >$. Some discussions are made in the last section.

II. Determining the Forms of Shift Operators from Physical Consideration

Let us focus on $| \psi_r >$ as shown in Eq. (4), and assume that $a(m_1, m_2, \cdots, m_r)$ are unknown coefficients at all. We shall determine the forms of shift operators from physical considerations. Firstly, we consider the raising operator that satisfies

$$Q_{r,r-1}^+ \mid \psi_{r-1} >=\mid \psi_r >, \tag{6}$$

namely, the raising operator $Q_{r,r-1}^+$ transforms the adjacent eigenstates specified by $r-1$ and r. An arbitrary state $\mid \psi_r >$ can be obtained by repeated applications of $Q_{r,r-1}^+$ to the given state $\mid \psi_0 >$: $\mid \psi_r >= Q_{r,r-1}^+ Q_{r-1,r-2}^+ \cdots Q_{2,1}^+ Q_{1,0}^+ \mid \psi_0 >$. Taking Eq. (6) and $H \mid \psi_r >= E_r \mid \psi_r >$ into account, one can easily verify Eq. (5).

Guided by the observation that $-i(\vec{S}_j \times \vec{S}_k)^- = S_j^- S_k^z - S_k^- S_j^z$ and

$$- i(\vec{S}_j \times \vec{S}_k)^- \mid \overset{j\,k}{\uparrow\uparrow}>= \frac{1}{2}(\mid \overset{j\,k}{\downarrow\uparrow}> - \mid \overset{j\,k}{\uparrow\downarrow}>), \quad -i(\vec{S}_j \times \vec{S}_k)^- \mid \overset{j\,k}{\downarrow\uparrow}>= \frac{1}{2} \mid \overset{j\,k}{\downarrow\downarrow}>, \tag{7}$$

i.e. $-i(\vec{S}_j \times \vec{S}_k)$ make a transition between singlet and triplet states of $\vec{S}_j \cdot \vec{S}_k$, we should expect to obtain that, after introducing the following unified raising operator

$$Q_{r,r-1}^+ = -i \sum_{j<k}^N W_{jk}^{(r)}(\vec{S}_j \times \vec{S}_k)^-, \quad W_{jk}^{(r)} = -W_{kj}^{(r)}, \tag{8}$$

Eq. (6) might be satisfied. Of course, other forms of raising operators such as $\mathcal{Q}_{r,r-1}^+ = \sum_{j=1}^N \alpha_j^{(r)} S_j^-$ can also be introduced. Direct calculations show that

$$- i \sum_{j<k}^N W_{jk}^{(r)}(\vec{S}_j \times \vec{S}_k)^- \mid \psi_{r-1} >= \sum_{j=1}^N \alpha_j^{(r)} S_j^- \mid \psi_{r-1} >=\mid \psi_r >, \tag{9}$$

where $\alpha_j^{(r)} = \frac{1}{2}\sum_{k\neq j}^N W_{jk}^{(r)}$. In other words, when $Q_{r,r-1}^+$ acts on $\mid \psi_{r-1} >$, it can be simplified to $\mathcal{Q}_{r,r-1}^+ = \sum_{j=1}^N \alpha_j^{(r)} S_j^-$, whose form is more simple. However, $\alpha_j^{(r)}$ is more complicate than $W_{jk}^{(r)}$, hence is much more difficult to determine than $W_{jk}^{(r)}$. It is the reason why we introduce $Q_{r,r-1}^+$ in the beginning, but not $\mathcal{Q}_{r,r-1}^+$. Eq. (9) means that $\mathcal{Q}_{r,r-1}^+$ has the same effect as $Q_{r,r-1}^+$, they are both the raising operators.

To find the forms of $Q_{r,r-1}^+$, we need to determine the forms of $W_{jk}^{(r)}$. In the following, we shall consider the cases $r = 1, 2, \cdots$ successively.

(a) $r = 1$. In this case, Eq. (6) becomes

$$\left[Q_{1,0}^+ = -i \sum_{j<k}^N W_{jk}^{(1)}(\vec{S}_j \times \vec{S}_k)^- \right] \mid \psi_0 >=\mid \psi_1 > . \tag{10}$$

Due to $\mid \psi_1 >= \sum_{m=1}^N a(m)\phi(m)$ and $-i(\vec{S}_j \times \vec{S}_k)^- \mid \overset{j\,k}{\uparrow\uparrow}>= \frac{1}{2}(\mid \overset{j\,k}{\downarrow\uparrow}> - \mid \overset{j\,k}{\uparrow\downarrow}>)$, by comparing the corresponding coefficients of $\phi(m)$ of the both sides of Eq. (10), we have

$$\frac{1}{2}[W_{m1}^{(1)} + \cdots + W_{m,m-1}^{(1)} + W_{m,m+1}^{(1)} + \cdots + W_{mN}^{(1)}] = a(m), \quad (m = 1, 2, \cdots). \tag{11}$$

Under periodic boundary condition $a(m+N) = a(m)$, it is well known that $\sum_{m=1}^N a(m) = 0$, so the solutions of Eq. (11) are

4

$$W_{jk}^{(1)} = \frac{2}{N}(a(j) - a(k)), \quad (j, k = 1, 2, \cdots, N). \tag{12}$$

(b) $r = 2$. In this case, Eq. (6) becomes

$$\left[Q_{2,1}^{+} = -i \sum_{j<k}^{N} W_{jk}^{(2)}(\vec{S}_j \times \vec{S}_k)^{-} \right] | \psi_1 >=| \psi_2 >, \tag{13}$$

where $| \psi_2 > = \sum_{m_1<m_2}^{N} a(m_1, m_2)\phi(m_1, m_2)$, and $\phi(m_1, m_2) = | \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\downarrow} \cdots >$ represents a spin state with two down-spins on the m_j-th ($j = 1, 2$) sites. After using $-i(\vec{S}_j \times \vec{S}_k)^{-} | \overset{j}{\downarrow}\overset{k}{\uparrow} > = \frac{1}{2} | \overset{j}{\downarrow}\overset{k}{\downarrow} >$ and comparing the corresponding coefficients of $\phi(m_1, m_2)$ of the both sides of Eq. (13), we have

$$\frac{1}{2}W_{m_1,m_2}^{(2)}[a(m_1) - a(m_2)] + \frac{1}{2} \sum_{m\neq m_1,m_2}^{N} [W_{m_2,m}^{(2)}a(m_1) + W_{m_1,m}^{(2)}a(m_2)] = a(m_1, m_2). \tag{14}$$

with $m_1, m_2 = 1, 2, \cdots, N$.

We now come to solve Eq. (14), i.e. to find the solutions of $W_{m_1,m_2}^{(2)}$. Because $a(m)$ depends only on m, we have

$$\left[\left(\frac{\partial a(m_1)}{\partial m_1}\right)^{-1} \frac{\partial}{\partial m_1}\right]a(m_1) = 1, \quad \left[\left(\frac{\partial a(m_1)}{\partial m_1}\right)^{-1} \frac{\partial}{\partial m_1}\right]a(m_2) = 0,$$

$$\left[\left(\frac{\partial a(m_2)}{\partial m_2}\right)^{-1} \frac{\partial}{\partial m_2}\right]a(m_2) = 1, \quad \left[\left(\frac{\partial a(m_2)}{\partial m_2}\right)^{-1} \frac{\partial}{\partial m_2}\right]a(m_1) = 0,$$

or in general

$$\left[\left(\frac{\partial a(m)}{\partial m}\right)^{-1} \frac{\partial}{\partial m}\right]a(j) = \delta_{jm}, \quad (m, j = 1, 2, \cdots, N), \tag{15}$$

where $m = x_m$ is the coordinate of the spin located on the $m-$th site of the lattice. From Eq. (15), one can obtain

$$\left[\left(\frac{\partial a(m)}{\partial m}\right)^{-1} \frac{\partial}{\partial m}\right] | \psi_1 >=\left[\left(\frac{\partial a(m)}{\partial m}\right)^{-1} \frac{\partial}{\partial m}\right] \sum_{m=1}^{N} a(m)\phi(m) = \phi(m), \tag{16}$$

hence, the action of the partial differential operator $\left(\frac{\partial a(m)}{\partial m}\right)^{-1} \frac{\partial}{\partial m}$ is clear, when it acts on $| \psi_1 >$, it will pick up the term $\phi(m)$ among $| \psi_1 >$.

From the above analysis, if we set

$$W_{j,k}^{(2)} = A(j, k) \left[\left(\frac{\partial a(j)}{\partial j}\right)^{-1} \frac{\partial}{\partial j} - \left(\frac{\partial a(k)}{\partial k}\right)^{-1} \frac{\partial}{\partial k}\right], \tag{17}$$

with

$$A(j,k) = A(k,j) = \begin{cases} a(j,k) & \text{if } j < k, \\ a(k,j) & \text{if } j > k, \end{cases} \tag{18}$$

then Eq. (14) is satisfied.

One can note that $W_{j,k}^{(2)}$ shown in Eq. (17) is a partial differential operator acting on the coefficients $a(j)$. In the following, we shall derive the form of shift operator $Q_{2,1}^+$ from physical consideration, so that physical picture transforming $| \psi_1 >$ into $| \psi_2 >$ can be seen clearly. In the former state, one of N spins is down, while in the latter, two of N spins are down. The crucial point is that, based on $| \psi_1 >$ in which there has already been one down-spin, how we can invert the second one so that it can be shifted to $| \psi_2 >$? Firstly, we must know clearly on which site the spin is down. The given state $\phi(m) = | \overset{1}{\uparrow}\overset{2}{\uparrow} \cdots \overset{m}{\downarrow} \cdots \overset{N}{\uparrow} >$ indicates that only on the m-th site the spin-state is down. With the help of the partial differential operator $\frac{1}{\partial a(m)/\partial m} \frac{\partial}{\partial m}$, the wanted term can be picked up from $| \psi_1 >$ [see Eq. (16)]. After this manipulation, we know clearly that on the $m-$th site the spin has already been down, thus the second down-spin can occur on the j-th $(j = 1, 2, \cdots, m-1, m+1, \cdots, N)$ sites, respectively. Meanwhile, $| \psi_2 >$ can be rewritten as

$$| \psi_2 >= \sum_{m_1 < m_2} a(m_1, m_2)\phi(m_1, m_2) = \frac{1}{2} \sum_{m=1}^{N} \Phi(m),$$

$$\Phi(m) = \sum_{j=1}^{m-1} a(j, m)\phi(j, m) + \sum_{k=m+1}^{N} a(m, k)\phi(m, k). \tag{19}$$

Introducing

$$T_m = \left[\sum_{j=1}^{m-1} T_{m \to (j,m)}^- + \sum_{k=m+1}^{N} T_{m \to (m,k)}^- \right],$$

since

$$[T_{m \to (j,m)}^- = a(j, m)(2i)(\vec{S}_j \times \vec{S}_m)^-]\phi(m) = a(j, m)\phi(j, m), \quad (j < m),$$

$$[T_{m \to (m,k)}^- = a(m, k)(-2i)(\vec{S}_m \times \vec{S}_k)^-]\phi(m) = a(m, k)\phi(m, k), \quad (k > m),$$

one can obtain $T_m\phi(m) = \Phi(m)$. Define $F_m = T_m \left(\frac{\partial a(m)}{\partial m} \right)^{-1} \frac{\partial}{\partial m}$, we then have

$$F_m | \psi_1 >= \sum_{j=1}^{m-1} a(j, m)\phi(j, m) + \sum_{k=m+1}^{N} a(m, k)\phi(m, k)$$

so that

$$\left(\sum_{m=1}^{N} F_m \right) | \psi_1 >= 2 | \psi_2 > . \tag{20}$$

6

Since $[\, Q_{2,1}^+ = -i \sum_{j<k}^N W_{jk}^{(2)} (\vec{S}_j \times \vec{S}_k)^- \,] \mid \psi_1 >= \mid \psi_2 >$, one then arrives at

$$Q_{2,1}^+ = \frac{1}{2} \sum_{m=1}^N F_m, \tag{21}$$

by identifying the coefficients of $(\vec{S}_j \times \vec{S}_k)^-$ of the both sides of Eq. (21), it leads to

$$W_{j,m}^{(2)} = a(j,m) \left[\left(\frac{\partial a(j)}{\partial j} \right)^{-1} \frac{\partial}{\partial j} - \left(\frac{\partial a(m)}{\partial m} \right)^{-1} \frac{\partial}{\partial m} \right],$$

$$W_{m,k}^{(2)} = a(m,k) \left[\left(\frac{\partial a(m)}{\partial m} \right)^{-1} \frac{\partial}{\partial m} - \left(\frac{\partial a(k)}{\partial k} \right)^{-1} \frac{\partial}{\partial k} \right], \tag{22}$$

i.e. they are nothing but the explicit expression of Eq. (17).

Therefore, the physical picture for the transformation from $\mid \psi_1 >$ to $\mid \psi_2 >$ is clear. This idea can be extended to generate $\mid \psi_r >$ for given $\mid \psi_{r-1} >$, for example, in the similar manner we can find the shift operator transforming $\mid \psi_2 >$ to $\mid \psi_3 >$.

(c) $r = 3$. Similar to (b), the eigenfunction $\mid \psi_3 >$ can be recast to $\mid \psi_3 >= \frac{1}{3} \sum_{m_1<m_2}^N \Phi(m_1, m_2)$, where

$$\Phi(m_1, m_2) = \sum_{j=1}^{m_1-1} a(j, m_1, m_2) \phi(j, m_1, m_2)$$

$$+ \sum_{k=m_1+1}^{m_2-1} a(m_1, k, m_2) \phi(m_1, k, m_2) + \sum_{l=m_2+1}^{N} a(m_1, m_2, l) \phi(m_1, m_2, l).$$

In $\mid \psi_2 >$ two of N spins are down, while in $\mid \psi_3 >$ there are three down-spins. Just like the case in (b), we introduce the partial differential operator $\left(\frac{\partial^2 a(m_1,m_2)}{\partial m_1 \partial m_2} \right)^{-1} \frac{\partial}{\partial m_1} \frac{\partial}{\partial m_2}$, which yields

$$\left(\frac{\partial^2 a(m_1, m_2)}{\partial m_1 \partial m_2} \right)^{-1} \frac{\partial}{\partial m_1} \frac{\partial}{\partial m_2} \mid \psi_2 >= \phi(m_1, m_2), \tag{23}$$

i.e. the term $\phi(m_1, m_2) = \mid \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\downarrow} \cdots >$ is picked up from $\mid \psi_2 >$, so that two down-spins on the m_1−th and m_2−th sites have been pre-set. Further, the third down-spin can occur on the j−th site:

$$(j = 1, 2, \cdots, m_1 - 1, m_1 + 1, m_1 + 2, \cdots, m_2 - 1, m_2 + 1, m_2 + 2, \cdots, N).$$

Introducing

$$T_{(m_1,m_2)} = \sum_{j=1}^{m_1-1} T_{(m_1,m_2)\to(j,m_1,m_2)}^-$$

$$+ \sum_{k=m_1+1}^{m_2-1} T_{(m_1,m_2)\to(m_1,k,m_2)}^- + \sum_{l=m_2+1}^{N} T_{(m_1,m_2)\to(m_1,m_2,l)}^-,$$

where

$$T^-_{(m_1,m_2)\to(j,m_1,m_2)} = a(j,m_1,m_2)\,(2i)[\,(\vec{S}_j \times \vec{S}_{m_1})^- + (\vec{S}_j \times \vec{S}_{m_2})^-\,],$$

$$T^-_{(m_1,m_2)\to(m_1,k,m_2)} = a(m_1,k,m_2)\,(2i)[\,-(\vec{S}_{m_1} \times \vec{S}_k)^- + (\vec{S}_k \times \vec{S}_{m_2})^-\,],$$

$$T^-_{(m_1,m_2)\to(m_1,m_2,l)} = a(m_1,m_2,l)\,(2i)[\,-(\vec{S}_{m_1} \times \vec{S}_l)^- - (\vec{S}_{m_2} \times \vec{S}_l)^-\,].$$

One can verify that $T_{(m_1,m_2)}\phi(m_1,m_2) = \Phi(m_1,m_2)$. Further, define

$$F_{(m_1,m_2)} = T_{(m_1,m_2)}\left(\frac{\partial^2 a(m_1,m_2)}{\partial m_1 \partial m_2}\right)^{-1}\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2},$$

we get

$$\left(\sum_{m_1<m_2}^{N} F_{(m_1,m_2)}\right) \mid \psi_2 >= 3 \mid \psi_3 >,$$

on the other hand, $Q^+_{3,2} \mid \psi_2 >= \mid \psi_3 >$, thus

$$Q^+_{3,2} = \frac{1}{3}\sum_{m_1<m_2}^{N} F_{(m_1,m_2)}, \tag{24}$$

by making comparison the coefficients of $(\vec{S}_j \times \vec{S}_k)^-$ of the both sides of Eq. (24), it yields

$$W^{(3)}_{j,k} = \frac{2}{3}\left\{\sum_{l\neq j,k}^{N} A(j,k,l)\left[\left(\frac{\partial^2 A(j,l)}{\partial j \partial l}\right)^{-1}\frac{\partial}{\partial j} - \left(\frac{\partial^2 A(k,l)}{\partial k \partial l}\right)^{-1}\frac{\partial}{\partial k}\right]\frac{\partial}{\partial l}\right\}. \tag{25}$$

with

$$A(j,k,l) = \begin{cases} a(j,k,l) & \text{if } j < k < l, \\ a(l,j,k) & \text{if } l < j < k, \\ a(j,l,k) & \text{if } j < l < k. \end{cases}$$

Thus the operator $Q^+_{3,2}$ shifting $\mid \psi_2 >$ to $\mid \psi_3 >$ is also found.

Making use of the similar analysis, one can obtain the general form of $Q^+_{r,r-1}$ with the solution of

$$W^{(r)}_{j,k} = \frac{2}{r}\left\{\sum_{l_1,l_2,\cdots,l_{r-2}\neq j,k}^{N} A(j,k,l_1,l_2,\cdots,l_{r-2})\times\right.$$

$$\left[\left(\frac{\partial^{r-1} A(j,l_1,l_2,\cdots,l_{r-2})}{\partial j \partial l_1 \partial l_2 \cdots \partial l_{r-2}}\right)^{-1}\frac{\partial}{\partial j} - \left(\frac{\partial^{r-1} A(k,l_1,l_2,\cdots,l_{r-2})}{\partial k \partial l_1 \partial l_2 \cdots \partial l_{r-2}}\right)^{-1}\frac{\partial}{\partial k}\right]\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\cdots\frac{\partial}{\partial l_{r-2}}\Bigg\},$$
$$\tag{26}$$

where $A(j,k,l_1,l_2,\cdots,l_{r-2})$ has the similar meaning as $A(j,k)$ shown in Eq. (18).

In (b) and (c) one can see that the actions of the partial differential operators $\left(\frac{\partial a(m)}{\partial m}\right)^{-1}\frac{\partial}{\partial m}$ and $\left(\frac{\partial^2 a(m_1,m_2)}{\partial m_1 \partial m_2}\right)^{-1}\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2}$ are picking up the terms $\phi(m)$ and $\phi(m_1,m_2)$ from $\mid \psi_1 >$ and $\mid \psi_2 >$, respectively. Since $\mid \psi_0 >$ has only one single term, we need not to introduce some partial differential operators to pick it up from $\mid \psi_0 >$. That is the reason why partial differential operators do not emerge in the coefficients $W^{(1)}_{j,k}$. In the

end of this section, we would like to point out that (i) raising operator $Q_{1,0}^+$ can be derived from physical consideration [see Appendix A]; (ii) The form of lowering operator $Q_{r-1,r}^-$ can also are obtained following the same spirit as **(b)** and **(c)** [see Appendix B].

III. Shift Operators for H_{xxx} and Energy Spectrum

To find the explicit result of $Q_{r,r-1}^+$ for H_{xxx}, we need to determine the unknown coefficients $W_{jk}^{(r)}$ or $a(m_1, m_2, \cdots, m_r)$. The direct calculation shows

$$[H_{xxx}, Q_{r,r-1}^+] = -J \sum_{j,k=j+1}^{N} W_{j,j+1}^{(r)} \{[(\vec{S}_{j-1} \times \vec{S}_j) \times \vec{S}_{j+1}]^-$$

$$+ \frac{1}{2}(S_j^- - S_{j+1}^-) - [\vec{S}_j \times (\vec{S}_{j+1} \times \vec{S}_{j+2})]^-\} - J \sum_{j,k \geq j+2}^{N} W_{j,k}^{(r)} \{[(\vec{S}_{j-1} \times \vec{S}_j) \times \vec{S}_k]^-$$

$$- [(\vec{S}_j \times \vec{S}_{j+1}) \times \vec{S}_k]^- + [\vec{S}_j \times (\vec{S}_{k-1} \times \vec{S}_k)]^- - [\vec{S}_j \times (\vec{S}_k \times \vec{S}_{k+1})]^-\}. \qquad (27)$$

In the following, only $| \psi_0 >$ is presumed known, its corresponding energy is $E_0 = 0$. Now we consider the cases $r = 1, 2, \cdots$ successively.

(a) $r = 1$. After acting Eq. (27) on $| \psi_0 >$, one obtains

$$[H_{xxx}, Q_{1,0}^+] | \psi_0 >= -iJ \sum_{j,k=j+1}^{N} \left[\left(\frac{W_{j+1,j+2}^{(1)} + W_{j-1,j}^{(1)}}{2W_{j,j+1}^{(1)}} - 1 \right) W_{j,j+1}^{(1)} (\vec{S}_j \times \vec{S}_{j+1})^- \right] | \psi_0 >$$

$$+ iJ \sum_{j,k=j+1}^{N} [(W_{j+1,j+2}^{(1)} S_{j+2}^- + W_{j-1,j}^{(1)} S_{j-1}^-)(\vec{S}_j \times \vec{S}_{j+1})^z] | \psi_0 >$$

$$- iJ \sum_{j,k \geq j+2}^{N} \left[\left(\frac{W_{j+1,k+1}^{(1)} + W_{j-1,k-1}^{(1)}}{2W_{j,k}^{(1)}} - 1 \right) W_{jk}^{(1)} (\vec{S}_j \times \vec{S}_k)^- \right] | \psi_0 >$$

$$+ iJ \sum_{j,k \geq j+2}^{N} W_{jk}^{(1)} [S_k^- (\vec{S}_{j-1} \times \vec{S}_j)^z - S_k^- (\vec{S}_j \times \vec{S}_{j+1})^z$$

$$- S_j^- (\vec{S}_{k-1} \times \vec{S}_k)^z + S_j^- (\vec{S}_k \times \vec{S}_{k+1})^z] | \psi_0 > . \qquad (28)$$

Since $(\vec{S}_j \times \vec{S}_k)^z | \psi_0 >= 0$, Eq. (28) then becomes

$$[H_{xxx}, Q_{1,0}^+] | \psi_0 >= -iJ \sum_{j<k}^{N} \left[\left(\frac{W_{j+1,k+1}^{(1)} + W_{j-1,k-1}^{(1)}}{2W_{jk}^{(1)}} - 1 \right) W_{jk}^{(1)} (\vec{S}_j \times \vec{S}_k)^- \right] | \psi_0 > . \quad (29)$$

Comparing Eq. (29) with the first of Eq. (5), $Q_{1,0}^+$ is a raising operator unless the factor $[W_{j+1,k+1}^{(1)} + W_{j-1,k-1}^{(1)}]/2W_{jk}^{(1)}$ is a real number and does not depend on j and k. If set

$$W_{jk}^{(1)} = \frac{2}{N}[a(j) - a(k)], \quad a(j) = \exp(ij\theta), \qquad (30)$$

9

Eq. (29) yields $[H_{xxx}, Q^+_{1,0}] \mid \psi_0 >= (\omega_{10} Q^+_{1,0}) \mid \psi_0 >$, where

$$\omega_{10} = J\left(\frac{W^{(1)}_{j+1,k+1} + W^{(1)}_{j-1,k-1}}{2W^{(1)}_{jk}} - 1\right) = J(\cos\theta - 1) \tag{31}$$

is the energy interval between E_1 and E_0. Under periodic boundary condition $a(m+N) = a(m)$, it is well known that

$$\theta = \frac{2\pi}{N}n; \quad n = \pm 1, \cdots, \pm(\frac{N}{2} - 1), \pm\frac{N}{2},$$

and $\sum_{m=1}^{N} a(m) = 0$. Since $E_0 = 0$, Eq. (31) leads to $E_1 = J(\cos\theta - 1)$. After acting $Q^+_{1,0}$ on $\mid \psi_0 >$, the next wavefunction $\mid \psi_1 >$ is obtained.

(b) $r = 2$. We set $\mid \psi_2 >= \sum_{j<k} a(j,k)\phi(j,k)$ with unknown expansion coefficients $a(j,k)$. The direct calculation shows

$$[H_{xxx}, Q^+_{2,1}] \mid \psi_1 >= J \sum_{j,k=j+1}^{N} \left[\frac{a(j-1,j+1) + a(j,j+2)}{2a(j,j+1)} - 2\right] a(j,j+1)\phi(j,j+1)$$

$$+ J \sum_{j,k\geq j+2}^{N} \left[\frac{a(j-1,k) + a(j+1,k) + a(j,k-1) + a(j,k+1)}{2a(j,k)} - 2\right] a(j,k)\phi(j,k)$$

$$- E_1(Q^+_{2,1} \mid \psi_1 >) - iJ \sum_{j<k}^{N} \left\{\left[\sum_{j=1}^{N}(\vec{S}_j \cdot \vec{S}_{j+1} - \frac{1}{4})\right](\vec{S}_j \times \vec{S}_k)^- W^{(2)}_{j,k} \sum_{m\neq j,k}^{N} a(m)\phi(m)\right\}. \tag{32}$$

To make $Q^+_{2,1}$ be a raising operator of H_{xxx}, we should require

$$Q^+_{2,1} \mid \psi_1 >=\mid \psi_2 >= \sum_{j<k} a(j,k)\phi(j,k), \tag{33}$$

$$W^{(2)}_{jk} \sum_{m\neq j,k}^{N} a(m)\phi(m) = 0, \tag{34}$$

and

$$\begin{aligned} E_2 &= \frac{a(j-1,k)+a(j+1,k)+a(j,k-1)+a(j,k+1)}{2a(j,k)} - 2 \\ &= \frac{a(j-1,j+1)+a(j,j+2)}{2a(j,j+1)} - 2. \end{aligned} \tag{35}$$

Since $a(m) = \exp(im\theta)$ is an exponential function, then one can find that

$$\frac{1}{i\theta}\frac{\partial}{\partial m}a(j) = \delta_{jm}a(m), \quad (m,j = 1,2,\cdots,N), \tag{36}$$

here $m = x_m$ is understood as the coordinate of the spin located on the m-th site of the lattice.

Due to Eq. (36), to make Eq. (34) be valid, one finds $W_{jk}^{(2)}$ can be the following solution

$$W_{jk}^{(2)} = \frac{1}{i\theta}a(j,k)\left[\frac{1}{a(j)}\frac{\partial}{\partial(m=j)} - \frac{1}{a(k)}\frac{\partial}{\partial(m=k)}\right], \quad (j < k) \tag{37}$$

with a still unknown coefficient $a(j,k)$, which will be determined by requiring that the two factor of the right-hand side of Eq. (35) are real numbers and do not depend on j and k. Obviously, if we choose $a(j,k)$ to be the usual Bethe ansatz, i.e.,

$$a(j,k) = Ce^{ij\theta_1}e^{ik\theta_2} + C'e^{ij\theta_2}e^{ik\theta_1}, \quad (j,k = 1,2,\cdots,N) \tag{38}$$

where C and C' are independent upon j and k. Eq. (38) yields

$$\frac{a(j-1,k) + a(j+1,k) + a(j,k-1) + a(j,k+1)}{2a(j,k)} = \cos\theta_1 + \cos\theta_2$$

for arbitrary C and C'. From Eq. (35) we should require

$$\frac{a(j-1,j+1) + a(j,j+2)}{2a(j,j+1)} = \cos\theta_1 + \cos\theta_2, \tag{39}$$

thus

$$\frac{C}{C'} = -\frac{1 - 2e^{i\theta_1} + e^{i(\theta_1+\theta_2)}}{1 - 2e^{i\theta_2} + e^{i(\theta_1+\theta_2)}}. \tag{40}$$

On the other hand, from the periodic condition $a(j,k) = a(k,j+N)$, we obtain

$$e^{iN\theta_1} = \frac{C}{C'} = e^{-iN\theta_2}. \tag{41}$$

Actually, Eqs.(40) and (41) can be rewritten in the following form

$$e^{iN\theta_1} = -e^{-i\Theta(\theta_1,\theta_2)}, \quad e^{iN\theta_2} = -e^{-i\Theta(\theta_2,\theta_1)}, \tag{42}$$

with

$$\Theta(\theta,\theta') = 2\arctan\left\{\frac{\sin\frac{\theta-\theta'}{2}}{\cos\frac{\theta+\theta'}{2} - \cos\frac{\theta-\theta'}{2}}\right\} = 2\arctan\left\{\frac{1}{2}\left[\cot\frac{\theta}{2} - \cot\frac{\theta'}{2}\right]\right\} \tag{43}$$

is an odd function, i.e., $\Theta(\theta,\theta') = -\Theta(\theta',\theta)$. From equation (42) we have

$$N\theta_1 = 2\pi I - \Theta(\theta_1,\theta_2), \quad N\theta_2 = 2\pi I' - \Theta(\theta_2,\theta_1), \tag{44}$$

where I and I' ($I' \neq I$) belong to the set $\{\pm\frac{1}{2},\pm\frac{3}{2},\cdots,\pm\frac{N-1}{2}\}$.

Thus we obtain two equations, i.e., (40) and (41), for the parameters θ_1 and θ_2, which determine the energy E_2:

$$E_2 = J(\cos\theta_1 + \cos\theta_2 - 2) = J\sum_{i=1}^{r=2}(\cos\theta_i - 1). \tag{45}$$

11

Eq. (44) is nothing but the usual Bethe Ansatz equation, obviously shift operators recover the results obtained through the usual Bethe Ansatz method.

If we select $C = \exp(i\frac{\phi_{12}}{2})$, $C' = \exp(-i\frac{\phi_{12}}{2})$, from Eq. (43) we have

$$2\cot\frac{\phi_{12}}{2} = \cot\frac{\theta_1}{2} - \cot\frac{\theta_2}{2}, \qquad -\pi \leq \phi_{12} \leq \pi. \tag{46}$$

After denoting $\phi_{21} = -\phi_{12}$, $m = m_1$, $k = m_2$, one obtains

$$a(m_1, m_2) = e^{i(\theta_1 m_1 + \theta_2 m_2 + \frac{\phi_{12}}{2})} + e^{(i\theta_1 m_2 + \theta_2 m_1 + \frac{\phi_{21}}{2})}, \tag{47}$$

whose form is quite convenient to be generalized to the general case. Moreover, since $a(j, k)$ is defined in the ordering $j < k$, to make $W_{jk}^{(2)} = -W_{kj}^{(2)}$, we rewrite $W_{jk}^{(2)}$ as

$$W_{jk}^{(2)} = \frac{1}{i\theta} A(j, k) \left[\frac{1}{a(j)} \frac{\partial}{\partial(m = j)} - \frac{1}{a(k)} \frac{\partial}{\partial(m = k)} \right], \tag{48}$$

where $A(j, k)$ is shown in Eq. (18).

(c) For general $r \geq 3$, This case is similar with case (b) by setting

$$a(m_1, m_2, \cdots, m_r) = \sum_{P=1}^{r!} \exp\left[i\left(\sum_{k=1}^{r} \theta_{P_k} m_k + \frac{1}{2} \sum \phi_{P_k, P_n} \right) \right]. \tag{49}$$

Substituting $Q_{r,r-1}^+$ with

$$W_{j,k}^{(r)} = \frac{2}{r} \frac{1}{i\theta_1} \frac{1}{i\theta_2} \cdots \frac{1}{i\theta_{r-1}} \left\{ \sum_{l_1, l_2, \cdots, l_{r-2} \neq j, k}^{N} A(j, k, l_1, l_2, \cdots, l_{r-2}) \right.$$

$$\left. \left[\frac{1}{A(j, l_1, l_2, \cdots, l_{r-2})} \frac{\partial}{\partial j} - \frac{1}{A(k, l_1, l_2, \cdots, l_{r-2})} \frac{\partial}{\partial k} \right] \frac{\partial}{\partial l_1} \frac{\partial}{\partial l_2} \cdots \frac{\partial}{\partial l_{r-2}} \right\}, \tag{50}$$

into the first of Eq. (5), and due to the periodic condition $a(m_2, \cdots, m_r, m_1 + N) = a(m_1, m_2, \cdots, m_r)$, we then have

$$E_r = J \sum_{m=1}^{r} (\cos\theta_m - 1), \tag{51}$$

where θ_m satisfies the Bethe equations

$$e^{iN\theta_m} = (-1)^{r-1} \prod_{k=1}^{r} \exp[-i\Theta(\theta_m, \theta_k)],$$

or

$$N\theta_m = 2\pi I_m - \sum_{k=1}^{r} \Theta(\theta_m, \theta_k), \quad (m \neq k) \tag{52}$$

for $m = 1, 2, \cdots, r$, and I_m runs over the r distinct integers $\{0, \pm 1, \cdots, \pm\frac{N}{2}\}$, for odd r, and $\{\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\frac{N-1}{2}\}$, for even r.

12

Lowering operator $Q_{r-1,r}^-$ can also be obtained by the similar way. Further more, it is worth mentioning that, like the case discussed by Bethe Ansatz method, the values of θ_m must be all distinct because, otherwise, $a(m_1, m_2, \cdots, m_r) = 0$ and $| \psi_r >= 0$. This is easily seen from equations (38) and (49) for $r = 2$.

IV. Shift Operators for H_{hs} and Energy Spectrum

To find the explicit result of $Q_{r,r-1}^+$ for H_{hs}, we need to determine the corresponding unknown coefficient $W_{jk}^{(r)}$, or $a(m_1, m_2, \cdots, m_r)$. The direct calculation shows

$$[H_{hs}, Q_{r,r-1}^+] = 4 \sum_{j<k}^N W_{j,k}^{(r)} \Bigg\{ \sum_{n=1, n\neq k-j}^{N-1} J_n [(\vec{S}_j \times \vec{S}_{j+n}) \times \vec{S}_k]^-$$

$$+ \sum_{n=1, n\neq N-(k-j)}^{N-1} J_n [\vec{S}_j \times (\vec{S}_k \times \vec{S}_{k+n})]^- - \frac{1}{2} J_{k-j} (S_j^- - S_k^-) \Bigg\}$$

$$= 4 \sum_{j<k}^N W_{j,k}^{(r)} \Bigg\{ \sum_{n=1, n\neq k-j}^{N-1} i J_n [(\vec{S}_j \times \vec{S}_{j+n})^- S_k^z - (\vec{S}_j \times \vec{S}_{j+n})^z S_k^-]$$

$$+ \sum_{n=1, n\neq N-(k-j)}^{N-1} i J_n [S_j^- (\vec{S}_k \times \vec{S}_{k+n})^z - S_j^z (\vec{S}_k \times \vec{S}_{k+n})^-] - \frac{1}{2} J_{k-j} (S_j^- - S_k^-) \Bigg\}. \quad (53)$$

where $J_n = J_{j,j+n} = J_0/\sin^2(n\pi/N)$, $J_{k-j} = J_0/\sin^2[(k-j)\pi/N]$. In the following, only $| \psi_0 >$ is presumed known, its corresponding energy is $E_0 = 0$. Now we consider the cases $r = 1, 2, \cdots$ successively.

(a) $r = 1$. After acting Eq. (53) on $| \psi_0 >$, and using $(\vec{S}_j \times \vec{S}_k)^z | \psi_0 >= 0$, one then obtains

$$[H_{hs}, Q_{1,0}^+] | \psi_0 >= -i \sum_{j<k}^N \Bigg[\Bigg(\sum_{n=1}^{N-1} \frac{W_{j-n,k-n}^{(1)}}{W_{j,k}^{(1)}} 2J_n - \sum_{n=1}^{N-1} 2J_n \Bigg) W_{j,k}^{(1)} (\vec{S}_j \times \vec{S}_k)^- \Bigg] | \psi_0 >$$

$$- [W_{k-N,k}^{(1)} J_{k-j} \phi(j) - W_{j,j}^{(1)} J_{k-j} \phi(k)]. \quad (54)$$

Comparing Eq. (54) with the first of Eq. (5), $Q_{1,0}^+$ is a raising operator unless the factor $W_{j-n,k-n}^{(1)}/W_{j,k}^{(1)}$ is a real number which does not depend on j and k, and

$$W_{k-N,k}^{(1)} = W_{j,j}^{(1)} = 0. \quad (55)$$

If we set $W_{jk}^{(1)} = 2[a(j) - a(k)]/N$, then Eq. (55) is satisfied, and

$$\frac{W_{j-n,k-n}^{(1)}}{W_{j,k}^{(1)}} = \frac{a(j-n) - a(k-n)}{a(j) - a(k)}, \quad (56)$$

which is required to be a real number and independent on j and k. Taking the periodic boundary condition $a(m + N) = a(m)$ into account, we have

13

$$a(m) = e^{im\pi}, \tag{57}$$

so that $W^{(1)}_{j-n,k-n}/W^{(1)}_{j,k} = e^{-in\pi} = (-1)^n$.

Denote $x = \sum_{n=1}^{N-1} J_n = \frac{J_0}{3}(N^2 - 1)$, $y = \sum_{n=1}^{N-1} (-1)^{n+1} J_n = \frac{J_0}{3}(\frac{N^2}{2} + 1)$, Eq. (54) yields $[H_{hs}, Q^+_{1,0}] \mid \psi_0 >= (\omega_{10} Q^+_{1,0}) \mid \psi_0 >$, where

$$\omega_{10} = 2\left(\sum_{n=1}^{N-1} (-1)^n J_n - \sum_{n=1}^{N-1} J_n\right) = -2(x+y) \tag{58}$$

is the energy interval between E_1 and E_0. Since $E_0 = 0$, then $E_1 = -2(x+y)$. After acting $Q^+_{1,0}$ on $\mid \psi_0 >$, the next wave function $\mid \psi_1 >$ is obtained.

(b) $r = 2$. We set $\mid \psi_2 >= \sum_{j<k} a(j,k)\phi(j,k)$ with unknown expansion coefficients $a(j,k)$. One can have

$$[H_{hs}, Q^+_{2,1}] \mid \psi_1 >= \sum_{j<k}^{N}\left[\sum_{n=1}^{N-1} 2J_n \frac{a(j-n,k) + a(j,k-n)}{a(j,k)} - 4\sum_{n=1}^{N-1} J_n\right.$$
$$\left. +4J_{k-j} - 2J_{k-j}\frac{a(k-N,k) + a(j,j)}{a(j,k)}\right] a(j,k)\phi(j,k) - E_1(Q^+_{2,1} \mid \psi_1 >)$$
$$- i\sum_{j<k}^{N}\left\{\left[\sum_{j=1}^{N} 4J_{k-j}\left(\vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_k - \frac{1}{4}\right)\right](\vec{S}_j \times \vec{S}_k)^- W^{(2)}_{j,k} \sum_{m\neq j,k}^{N} a(m)\phi(m)\right\}. \tag{59}$$

To make $Q^+_{2,1}$ be a raising operator of H_{hs}, we should require

$$Q^+_{2,1} \mid \psi_1 >= \mid \psi_2 >= \sum_{j<k} a(j,k)\phi(j,k), \tag{60}$$

$$W^{(2)}_{jk} \sum_{m\neq j,k}^{N} a(m)\phi(m) = 0, \tag{61}$$

and

$$\begin{aligned}E_2 &= \sum_{n=1}^{N-1} 2J_n \frac{a(j-n,k)+a(j,k-n)}{a(j,k)} - 4\sum_{n=1}^{N-1} J_n \\ &\quad + 4J_{k-j} - 2J_{k-j}\frac{a(k-N,k)+a(j,j)}{a(j,k)}.\end{aligned} \tag{62}$$

To make Eqs.(60) and (61) be valid, one can set

$$W^{(2)}_{j,k} = a(j,k)\left[\left(\frac{\partial a(j)}{\partial j}\right)^{-1}\frac{\partial}{\partial j} - \left(\frac{\partial a(k)}{\partial k}\right)^{-1}\frac{\partial}{\partial k}\right], \quad (j < k)$$

with a still unknown coefficient $a(j,k)$, which will be determined by requiring that the right-hand side of Eq. (62) be a real number that does not depend on j and k.

On one hand, since $\phi(m_1, m_2)$ represents a spin state with two down-spins on the m_i-th $(i = 1, 2)$ sites of the lattice, and there is only one spin on each site, $\phi(j,j)$ and $\phi(k,k)$ would not have physical meanings, hence we should require

14

$$a(j, j) = 0, \quad a(k - N, k) = a(k, k) = 0. \tag{63}$$

On the other hand, $-4\sum_{n=1}^{N-1} J_n = -4x$ does not depend on j and k, from Eq. (62) one requires

$$E_2' = \sum_{n=1}^{N-1} 2J_n \frac{a(j - n, k) + a(j, k - n)}{a(j, k)} + 4J_{k-j} \tag{64}$$

be a real number that does not depend on j and k. Now let us come to find the coefficients $a(j, k)$ satisfying the requirements Eqs.(63) and (64).

(i) Like the case in the XXX-Heisenberg model, in the beginning, we set the Ansatz

$$a(j, k) = Ce^{ij\theta_1}e^{ik\theta_2} + C'e^{ij\theta_2}e^{ik\theta_1}, \quad (j, k = 1, 2, \cdots, N), \tag{65}$$

where C and C' are independent upon j and k. Eq. (65) yields

$$\frac{a(j - n, k) + a(j, k - n)}{a(j, k)} = e^{-in\theta_1} + e^{-in\theta_2} \tag{66}$$

for arbitrary C and C'. After substituting Eq. (66) into the E_2' as shown in Eq. (64), one will find that the first term of E_2', i.e., $\sum_{n=1}^{N-1} 2J_n[a(j - n, k) + a(j, k - n)]/a(j, k)$ is independent upon j and k, while the second term of E_2', i.e., $4J_{k-j}$ depends on j and k, so the Ansatz shown in Eq. (65) does not satisfy the requirement (64). Based on this observation, the factor $[a(j - n, k) + a(j, k - n)]/a(j, k)$ will have to depend on j and k, so that the summation $\sum_{n=1}^{N-1} 2J_n[a(j - n, k) + a(j, k - n)]/a(j, k)$ can contribute a term $-4J_{k-j}$ to cancel the second term of E_2'.

(ii) We modify the Ansatz as following:

$$a(j, k) = C[e^{ij\theta_1}e^{ik\theta_2} + \lambda] + C'[e^{ij\theta_2}e^{ik\theta_1} + \lambda'], \tag{67}$$

where C, C', λ and λ' do not depend on j and k. Eq. (67) yields

$$\frac{a(j - n, k) + a(j, k - n)}{a(j, k)} = e^{-in\theta_1} + e^{-in\theta_2} + \frac{2(C\lambda + C'\lambda')[1 - (e^{-in\theta_1} + e^{-in\theta_2})/2]}{C[e^{ij\theta_1}e^{ik\theta_2} + \lambda] + C'[e^{ij\theta_2}e^{ik\theta_1} + \lambda']}. \tag{68}$$

From Eq. (67) one obtains

$$a(j, j) = C[e^{ij(\theta_1+\theta_2)} + \lambda] + C'[e^{ij(\theta_1+\theta_2)} + \lambda'] = (C + C')e^{ij(\theta_1+\theta_2)} + (C\lambda + C'\lambda'), \tag{69}$$

since $a(j, j) = 0$ for arbitrary j, we then have the following two kinds of possibilities:

(ii-1): $C = -C', \quad C\lambda + C'\lambda' = 0.$

However, $C\lambda + C'\lambda' = 0$ will let the factor $[a(j-n,k) + a(j,k-n)]/a(j,k)$ do not depend on j and k, so we eliminate this possibility.

(ii-2):

$$\lambda = \lambda' = -1, \quad \theta_1 + \theta_2 = 0, \quad \text{or} \quad 2\pi \times \text{integer}, \tag{70}$$

so that

$$a(j,k) = C[e^{-i(k-j)\theta_1} - 1] + C'[e^{i(k-j)\theta_1} - 1],$$

$$\frac{a(j-n,k) + a(j,k-n)}{a(j,k)} = 2\cos(n\theta_1) + \frac{-2(C+C')[1-\cos(n\theta_1)]}{a(j,k)}. \tag{71}$$

In order to cancel the second term of E_2' (i.e., $4J_{k-j}$), we should require

$$a(j,k) \quad \propto \quad \frac{1}{4J_{k-j}} \quad \propto \quad \sin^2[(k-j)\pi/N], \tag{72}$$

or

$$\frac{C}{C'}[e^{-i(k-j)\theta_1} - 1] + [e^{i(k-j)\theta_1} - 1] \quad \propto \quad e^{i2(k-j)\pi/N} + e^{-i2(k-j)\pi/N} - 2. \tag{73}$$

On the other hand, the periodic condition $a(j,k) = a(k, j+N)$ yields

$$e^{iN\theta_1} = \frac{C}{C'} = e^{-iN\theta_2}. \tag{74}$$

thus one obtains

$$C = C', \quad \theta_1 = -\theta_2 = 2\pi/N, \tag{75}$$

which leads to

$$\frac{a(j-n,k) + a(j,k-n)}{a(j,k)} = 2\left[1 - \sin^2(n\pi/N) + \frac{\sin^2(n\pi/N)}{\sin^2[(k-j)\pi/N]}\right].$$

The summation

$$\sum_{n=1}^{N-1} 4J_n \sin^2(n\pi/N)/\sin^2[(k-j)\pi/N] = \sum_{n=1}^{N-1} 4J_0/\sin^2[(k-j)\pi/N] = (N-1)4J_{k-j}$$

cannot cancel the second term of E_2' (i.e. $4J_{k-j}$). However, it is easy to observe that the summation $\sum_{n=1}^{N-1}(-1)^n 4J_0/\sin^2[(k-j)\pi/N] = -4J_{k-j}$ can reach the purpose. This observation renders us to modify again the Ansatz as following:

(iii) We set the Ansatz:

$$a(j,k) = e^{i\pi(j+k)}\{C[e^{ij\theta_1}e^{ik\theta_2} + \lambda] + C'[e^{ij\theta_2}e^{ik\theta_1} + \lambda']\}, \tag{76}$$

or

$$a(j,k) = e^{i\pi(j+k)}\sin^2[(k-j)\pi/N], \quad (j,k = 1,2,\cdots,N) \tag{77}$$

16

whose form is quite convenient to be generalized to the general case for $r > 2$. Eq. (77) yields directly $a(k - N, k) = a(j, j) = 0$, and

$$\frac{a(j - n, k) + a(j, k - n)}{a(j, k)} = 2e^{-i\pi n}\left[1 - 2\sin^2(n\pi/N) + \frac{\sin^2(n\pi/N)}{\sin^2[(k - j)\pi/N]}\right]. \tag{78}$$

From Eq. (62) we obtain

$$E_2 = -4(x + y) + 8J_0. \tag{79}$$

After acting $Q_{2,1}^+$ on $\mid \psi_1 >$, the next wave function $\mid \psi_2 >$ is obtained.

(c) $r = 3$. We set

$$\mid \psi_3 >= \sum_{m_1 < m_2 < m_3} a(m_1, m_2, m_3)\phi(m_1, m_2, m_3) \tag{80}$$

with unknown expansion coefficients $a(m_1, m_2, m_3)$. The direct calculation shows

$$[H_{hs}, Q_{3,2}^+] \mid \psi_2 >=$$

$$\sum_{m_1 < m_2 < m_3}\Bigg\{ \sum_{n=1}^{N-1} 2J_n \frac{a(m_1 - n, m_2, m_3) + a(m_1, m_2 - n, m_3) + a(m_1, m_2, m_3 - n)}{a(m_1, m_2, m_3)}$$

$$-6\sum_{n=1}^{N-1} J_n + 4(J_{m_2-m_1} + J_{m_3-m_1} + J_{m_3-m_2})$$

$$-\frac{2J_{m_2-m_1}[a(m_1, m_1, m_3) + a(m_2 - N, m_2, m_3)]}{a(m_1, m_2, m_3)}$$

$$-\frac{2J_{m_3-m_1}[a(m_1, m_2, m_1) + a(m_3 - N, m_2, m_3)]}{a(m_1, m_2, m_3)}$$

$$-\frac{2J_{m_3-m_2}[a(m_1, m_2, m_2) + a(m_1, m_3 - N, m_3)]}{a(m_1, m_2, m_3)}\Bigg\}a(m_1, m_2, m_3)\phi(m_1, m_2, m_3)$$

$$-E_2(Q_{3,2}^+ \mid \psi_2 >)$$

$$- i\sum_{j<k}^{N}\Bigg\{\Bigg[\sum_{j=1}^{N} 4J_{k-j}\left(\vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_k - \frac{1}{4}\right)\Bigg](\vec{S}_j \times \vec{S}_k)^- W_{j,k}^{(3)} \sum_{m_1,m_2\neq j,k}^{N} a(m_1, m_2)\phi(m_1, m_2)\Bigg\}. \tag{81}$$

To make $Q_{3,2}^+$ be a raising operator of H_{hs}, we should require

$$Q_{3,2}^+ \mid \psi_2 >=\mid \psi_3 >= \sum_{m_1 < m_2 < m_3} a(m_1, m_2, m_3)\phi(m_1, m_2, m_3), \tag{82}$$

$$W_{j,k}^{(3)} \sum_{m_1,m_2\neq j,k}^{N} a(m_1, m_2)\phi(m_1, m_2) = 0 \tag{83}$$

17

and

$$E_3 = \sum_{n=1}^{N-1} 2J_n \frac{a(m_1 - n, m_2, m_3) + a(m_1, m_2 - n, m_3) + a(m_1, m_2, m_3 - n)}{a(m_1, m_2, m_3)}$$

$$-6 \sum_{n=1}^{N-1} J_n + 4(J_{m_2-m_1} + J_{m_3-m_1} + J_{m_3-m_2})$$

$$-\frac{2J_{m_2-m_1}[a(m_1, m_1, m_3) + a(m_2 - N, m_2, m_3)]}{a(m_1, m_2, m_3)}$$

$$-\frac{2J_{m_3-m_1}[a(m_1, m_2, m_1) + a(m_3 - N, m_2, m_3)]}{a(m_1, m_2, m_3)}$$

$$-\frac{2J_{m_3-m_2}[a(m_1, m_2, m_2) + a(m_1, m_3 - N, m_3)]}{a(m_1, m_2, m_3)} \tag{84}$$

To make Eqs.(82) and (83) be valid, one can set

$$W_{j,k}^{(3)} = \frac{2}{3}\left\{ \sum_{l \neq j,k}^{N} A(j,k,l)\left[\left(\frac{\partial^2 A(j,l)}{\partial j \partial l}\right)^{-1}\frac{\partial}{\partial j} - \left(\frac{\partial^2 A(k,l)}{\partial k \partial l}\right)^{-1}\frac{\partial}{\partial k}\right]\frac{\partial}{\partial l} \right\}.$$

with a still unknown coefficient $a(j, k, l)$, which will be determined by requiring that the right-hand side of Eq. (84) be a real number that does not depend on j and k. If we choose

$$a(m_1, m_2, m_3) = e^{i\pi \sum_{i=1}^{3} m_i} \prod_{i<j} \sin^2(\pi(m_j - m_i)/N), \quad (m_1, m_2, m_3 = 1, 2, \cdots, N) \tag{85}$$

it yields

$$a(m_1, m_1, m_3) = a(m_2 - N, m_2, m_3) = a(m_3 - N, m_2, m_3) = 0, \quad etc.$$

Denote

$$\lambda_1 = (m_3 - m_2)\pi/N, \quad \lambda_2 = (m_2 - m_1)\pi/N, \quad \lambda_3 = (m_3 - m_1)\pi/N = \lambda_1 + \lambda_2,$$

we then have

$$\frac{a(m_1 - n, m_2, m_3) + a(m_1, m_2 - n, m_3) + a(m_1, m_2, m_3 - n)}{a(m_1, m_2, m_3)} = e^{-in\pi}\bigg\{$$

$$3[1 - 2\sin^2(n\pi/N)]^2 + 2[1 - 2\sin^2(n\pi/N)]\sin^2(n\pi/N)\sum_{i=1}^{3}\left(\frac{1}{\sin^2(\lambda_i)}\right)$$

$$+\sin^4(n\pi/N)\left(\frac{1}{\sin^2(\lambda_1)\sin^2(\lambda_2)} + \frac{1}{\sin^2(\lambda_1)\sin^2(\lambda_3)} + \frac{1}{\sin^2(\lambda_2)\sin^2(\lambda_3)}\right)$$

$$+\sin^2(2n\pi/N)[\cot(\lambda_1)\cot(\lambda_3) + \cot(\lambda_2)\cot(\lambda_3) - \cot(\lambda_1)\cot(\lambda_2)]\bigg\}. \tag{86}$$

Since

$$\cot(\lambda_1)\cot(\lambda_3) + \cot(\lambda_2)\cot(\lambda_3) - \cot(\lambda_1)\cot(\lambda_2) = -1,$$

$$\sum_{n=1}^{N-1} e^{-in\pi} \sin^2(n\pi/N) = 0, \quad \sum_{n=1}^{N-1} e^{-in\pi} \cos^2(n\pi/N) = -1, \quad \sum_{n=1}^{N-1} e^{-in\pi} = -1, \qquad (87)$$

from Eq. (84) one obtains

$$E_3 = -6(x+y) + 32J_0. \qquad (88)$$

For general $r \geq 4$, by setting

$$\mid \psi_r >= \sum_{m_1 < m_2 < \cdots < m_r} a(m_1, m_2, \cdots, m_r)\phi(m_1, m_2, \cdots, m_r),$$

$$a(m_1, m_2, \cdots, m_r) = e^{i\pi \sum_{j=1}^{r} m_j} \prod_{i<j} \sin^2(\pi(m_j - m_i)/N), \qquad (89)$$

and making use of the similar analysis, one can obtain shift operators $Q_{r,r-1}^+$ and $Q_{r-1,r}^-$, and the energy E_r corresponding to $\mid \psi_r >$ is

$$E_r = -2r(x+y) + 4r(r-1)J_0 + \frac{4}{3}r(r-1)(r-2)J_0. \qquad (90)$$

V. Discussion and Conclusion

A standard quantum mechanical transition problem in general has the following format. The first quantity we must have is a Hamiltonian $\mathcal{H}$, which can be divided as $\mathcal{H} = H_0 + H_I$, where for some region of coordinate space or time H_I can be neglected. Secondly, when H_I is neglected, it is meaningful of speak of the energy levels and corresponding states of the free Hamiltonian H_0 between which the transitions take place. These transitions are induced by the interaction H_I. Experiments can detect the frequencies (i.e. the energy intervals) satisfying the Bohr frequency condition $\omega_{nm} = E_n - E_m$. The energy spectrum of H_0 can be determined from experiments is owing to the existence of the external interaction H_I. The physical nature of the external factor, which causes the quantum transition of the microparticles is arbitrary. In particular, it may be the interaction of the microparticles with electromagnetic radiation. Typical examples can be seen in a hydrogen atom or a harmonic oscillator, where a transition from one stationary state to another is realized by an electric dipole moment. The dipole moment operator $\hat{\mathbf{d}}$ of any atom is expressible as a sum of raising and lowering operators $\hat{\mathcal{L}}(n,m)$ between states $\mid \psi_m >$ and $\mid \psi_n >$ [24]: $\hat{\mathbf{d}} = \sum_{n,m} \mathbf{d}_{nm} \hat{\mathcal{L}}(n,m)$, where

$$\hat{\mathcal{L}}^+(n,m) = Q_{n,n-1}^+ Q_{n-1,n-2}^+ \cdots Q_{m+2,m+1}^+ Q_{m+1,m}^+,$$

$$\hat{\mathcal{L}}^-(m,n) = Q_{m,m+1}^- Q_{m+1,m+2}^- \cdots Q_{n-2,n-1}^- Q_{n-1,n}^-, \quad (n>m).$$

Usually, in a hydrogen atom or a harmonic oscillator, the dipole moment operator is the coordinate $\mathbf{r}$ or the momentum $\mathbf{p}$ of the particle, and the interaction H_I is expressed by the scalar product of the dipole moment operator and the external field. We would like to extend this kind of dipole transition mechanism to the XHM or HSM so that its energy spectrum might be detected from experiments.

Fistly, we take the H_{xxx} or H_{hs} as the free Hamiltonian H_0. Secondly, we write the interaction as $H_I(t) = \hat{\mathbf{d}} \cdot \mathbf{B}(t)$, where $\mathbf{B}(t) = \sum_\lambda [\, \mathbf{e}(\lambda)\mathcal{E}_\lambda^{(+)}(t) + \mathbf{e}^*(\lambda)\mathcal{E}_\lambda^{(-)}(t) \,]$ is a time-dependent magnetic field, and

$$\mathcal{E}_\lambda^{(+)}(t) = \mathcal{E}_\lambda(t)\exp(-i\omega_\lambda t), \quad \mathcal{E}_\lambda^{(-)}(t) = \mathcal{E}_\lambda^*(t)\exp(+i\omega_\lambda t).$$

Hence, the time-dependent Schrödinger equation is

$$i\frac{\partial}{\partial t} \mid \Psi(t) >= (H_0 + \hat{\mathbf{d}} \cdot \mathbf{B}(t)) \mid \Psi(t) > .$$

The general state $\mid \Psi(t) >$ is written as an expansion

$$\mid \Psi(t) >= \sum_{n=1} C_n(t) \mid \psi_n > \exp(-iE_n t),$$

where $\mid \psi_n >$'s are eigenstates of H_0. We thereby obtain a set couple equations

$$-i\tfrac{d}{dt}C_n(t) \;\; = \;\; \sum_k \sum_\lambda \mathbf{d}_{nk} \cdot \mathbf{e}(\lambda)\mathcal{E}_\lambda \exp[-i(\omega_\lambda - \omega_{nk})t]C_k(t)$$
$$+ \sum_k \sum_\lambda \mathbf{d}_{nk} \cdot \mathbf{e}^*(\lambda)\mathcal{E}_\lambda^* \exp[+i(\omega_\lambda - \omega_{nk})t]C_k(t),$$

with $\omega_{nk} = E_n - E_k$ is the Bohr transition frequency. Obviously, when $\omega_\lambda = \omega_{nk}$, the magnetic resonance phenomena would happen, thus the energy intervals of the spin chain might be detected.

Starting from the ferromagnetic "vacuum" state $\mid \psi_0 >$, if the raising operators are acted for enough times, then for even-spins antiferromagnetic XHM (or HSM), it will reach the ground state. The corresponding ground state energy was first calculated by Hulthén using Bethe's method [25]. The ground state is a singlet with total spin $S_T = 0$ (for N =even integer), therefore the number of spin-deviates in the ground state is $r = N/2$ (The proof that the total spin is indeed minimal in the ground state is found in [26]). des Cloiseaux and Pearson (dCP) were the first to study the elementary excitations [27], which they interpreted as spin-wavelike states with $S_T = 1$. It was later shown by Faddeev and Takhtajan [28] that the natural excitations (spinons) actually have $S_T = 1/2$, and hence fermions. The underlying excitations occur only in pairs [28]. The dCP states are now understood to be a superposition of two spinons, one of which carries zero momentum. If the energy spectrum is detected by the dipole transition mechanism, people might ask: What are the first and the second excited states? And what are their degeneracies? These problems are still open and under investigation. In conclusion, we have developed a unified

treatment for the XXX-Heisenberg model and the H_2 in Haldane-Shastry model using shift operators, based on which the energy spectra of the models are determined. Furthermore, it is also interesting and significant to extend the shift operator approach to the Hubbard model [29][30] and the generalized Bethe ansatz [31] in subsequent investigations.

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APPENDIX A: Physical Picture of Shifting $\mid \psi_0 >$ to $\mid \psi_1 >$

Because

$$-i(\vec{S}_j \times \vec{S}_k)^- \mid \overset{jk}{\uparrow\uparrow}>= \frac{1}{2}(\mid \overset{jk}{\downarrow\uparrow}> - \mid \overset{jk}{\uparrow\downarrow}>), \quad (S_j^- + S_k^-)\mid \overset{jk}{\uparrow\uparrow}>= (\mid \overset{jk}{\downarrow\uparrow}> + \mid \overset{jk}{\uparrow\downarrow}>), \qquad (91)$$

so that

$$\mid \overset{jk}{\downarrow\uparrow}>= [-i(\vec{S}_j \times \vec{S}_k)^- + \frac{1}{2}(S_j^- + S_k^-)] \mid \overset{jk}{\uparrow\uparrow}>,$$

$$\mid \overset{jk}{\uparrow\downarrow}>= [i(\vec{S}_j \times \vec{S}_k)^- + \frac{1}{2}(S_j^- + S_k^-)] \mid \overset{jk}{\uparrow\uparrow}> . \qquad (92)$$

Due to Eq. (92), one finds the transformation

$$\mid \overset{1\,2}{\uparrow\uparrow} \cdots \overset{m}{\uparrow} \cdots \overset{N}{\uparrow}>\Longrightarrow a(m)\phi(m) = a(m) \mid \overset{1\,2}{\uparrow\uparrow} \cdots \overset{m}{\downarrow} \cdots \overset{N}{\uparrow}> \quad (m = 1, 2, \cdots, N),$$

can be realized by the operator:

$$T_{0\to m}^- = \frac{a(m)}{N-1}\left\{ \sum_{j=1}^{m-1}\left[i(\vec{S}_j \times \vec{S}_m)^- + \frac{1}{2}(S_j^- + S_m^-)\right] + \sum_{j=m+1}^{N}\left[-i(\vec{S}_m \times \vec{S}_j)^- + \frac{1}{2}(S_m^- + S_j^-)\right]\right\}. \qquad (93)$$

Define

$$F_0 = \sum_{m=1}^{N} T_{0\to m}^- =$$

$$-i\frac{1}{N-1}\sum_{j<k}^{N}[a(j) - a(k)](\vec{S}_j \times \vec{S}_k)^- + \frac{1}{N-1}\sum_{m=1}^{N}\left[\frac{N-2}{2}a(m) + \frac{1}{2}\sum_{k=1}^{N}a(k)\right]S_m^-, \quad (94)$$

one gets

$$F_0 \mid \psi_0 >= \sum_{m=1}^{N} a(m)\phi(m) =\mid \psi_1 > \qquad (95)$$

Using

$$\sum_{m=1}^{N} a(m) = 0, \quad \sum_{m=1}^{N} a(m)S_m^- \mid \psi_0 >=\mid \psi_1 >, \qquad (96)$$

from Eq. (94) and Eq. (95) we then have

$$-i\frac{1}{N-1}\sum_{j<k}^{N}[a(j) - a(k)](\vec{S}_j \times \vec{S}_k)^- \mid \psi_0 >=\left(1 - \frac{1}{2}\frac{N-2}{N-1}\right) \mid \psi_1 >, \qquad (97)$$

thus

$$-i\frac{2}{N}\sum_{j<k}^{N}[a(j) - a(k)](\vec{S}_j \times \vec{S}_k)^- \mid \psi_0 >=\mid \psi_1 >, \qquad (98)$$

on the other hand

$$\left[Q_1^- = -i\sum_{j<k}^{N}W_{jk}^{(1)}(\vec{S}_j \times \vec{S}_k)^-\right] \mid \psi_0 >=\mid \psi_1 >, \qquad (99)$$

23

by comparing the coefficients of $(\vec{S}_j \times \vec{S}_k)^-$ of the left-hand sides of Eqs.(98) and (99) , it leads to

$$W_{jk}^{(1)} = \frac{2}{N}(a(j) - a(k)), \quad (j,k = 1, 2, \cdots, N) \tag{100}$$

which is nothing but Eq. (12). Thus the physical picture for the transformation from $\mid \psi_0 >$ to $\mid \psi_1 >$ is also clear.

APPENDIX B: The Forms of Lowering Operators

Because $i(\vec{S}_j \times \vec{S}_k)^+ = S_j^+ S_k^z - S_k^+ S_j^z$, one can calculate that

$$i(\vec{S}_j \times \vec{S}_k)^+ \mid \overset{jk}{\uparrow\downarrow}>= -\frac{1}{2} \mid \overset{jk}{\uparrow\uparrow}>; \quad i(\vec{S}_j \times \vec{S}_k)^+ \mid \overset{jk}{\downarrow\uparrow}>= \frac{1}{2} \mid \overset{jk}{\uparrow\uparrow}> . \tag{101}$$

We set the lowering operator $Q_{0,1}^-$ is

$$\left[Q_{0,1}^- = i\sum_{j<k}^{N} W_{jk}'^{(1)}(\vec{S}_j \times \vec{S}_k)^+ \right] \mid \psi_1 >= \mid \psi_0 >, \quad W_{jk}'^{(1)} = -W_{kj}'^{(1)}, \tag{102}$$

from which we obtain the equation

$$\frac{1}{2}\sum_{m=1}^{N}\left[\left(-\sum_{j=1}^{m-1} W_{jm}'^{(1)} + \sum_{k=m+1}^{N} W_{mk}'^{(1)} \right)a(m) \right] = 1, \tag{103}$$

whose solutions are

$$W_{j,k}'^{(1)} = \frac{2}{N(N-1)}\left[\left(\frac{\partial a(j)}{\partial j}\right)^{-1}\frac{\partial}{\partial j} - \left(\frac{\partial a(k)}{\partial k}\right)^{-1}\frac{\partial}{\partial k} \right]. \tag{104}$$

Because

$$i(\vec{S}_j \times \vec{S}_k)^+ \mid \overset{jk}{\downarrow\downarrow}>= \frac{1}{2}(\mid \overset{jk}{\downarrow\uparrow}> - \mid \overset{jk}{\uparrow\downarrow}>), \quad (S_j^+ + S_k^+) \mid \overset{jk}{\downarrow\downarrow}>= (\mid \overset{jk}{\downarrow\uparrow}> + \mid \overset{jk}{\uparrow\downarrow}>), \tag{105}$$

so that

$$\mid \overset{jk}{\downarrow\uparrow}>=\left[i(\vec{S}_j \times \vec{S}_k)^+ + \frac{1}{2}(S_j^+ + S_k^+) \right] \mid \overset{jk}{\downarrow\downarrow}>,$$

$$\mid \overset{jk}{\uparrow\downarrow}>=\left[- i(\vec{S}_j \times \vec{S}_k)^+ + \frac{1}{2}(S_j^+ + S_k^+) \right] \mid \overset{jk}{\downarrow\downarrow}> . \tag{106}$$

Therefore, based on Eq. (106) there would be some differences in achieving lowering operators $Q_{r-1,r}^-$ when $r > 1$. In the position, we set

$$\left[Q_{r-1,r}^- = i\sum_{j<k}^{N} W_{jk}'^{(r)}(\vec{S}_j \times \vec{S}_k)^+ + \sum_{j=1}^{N} \lambda_j^{(r)} S_j^+ \right] \mid \psi_r >= \mid \psi_{r-1} >, \quad W_{jk}'^{(r)} = -W_{kj}'^{(r)}, \tag{107}$$

which is different from the raising operator by a translation term $\sum_{j=1}^{N} \lambda_j^{(r)} S_j^+$.

For instance, when $r = 2$, since

$$| \psi_2 > = \sum_{m_1 < m_2} a(m_1, m_2) | \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\downarrow} \cdots >,$$

$$| \psi_1 > = \frac{1}{N-1} \sum_{m_1 < m_2} [\, a(m_1) | \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\uparrow} \cdots > + a(m_2) | \cdots \overset{m_1}{\uparrow} \cdots \overset{m_2}{\downarrow} \cdots > \,],$$

owing to Eq. (106), one finds the transformation

$$a(m_1, m_2) | \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\downarrow} \cdots > \Longrightarrow a(m_1) | \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\uparrow} \cdots >$$

can be realized by the operator:

$$T^-_{(m_1, m_2) \to m_1} = \frac{a(m_1)}{a(m_1, m_2)} \left[i(\vec{S}_{m_1} \times \vec{S}_{m_2})^+ + \frac{1}{2}(S_{m_1}^+ + S_{m_2}^+) \right], \tag{108}$$

and the transformation

$$a(m_1, m_2) | \cdots \overset{m_1}{\downarrow} \cdots \overset{m_2}{\downarrow} \cdots > \Longrightarrow a(m_2) | \cdots \overset{m_1}{\uparrow} \cdots \overset{m_2}{\downarrow} \cdots >$$

is realized by the operator:

$$T^-_{(m_1, m_2) \to m_2} = \frac{a(m_2)}{a(m_1, m_2)} \left[-i(\vec{S}_{m_1} \times \vec{S}_{m_2})^+ + \frac{1}{2}(S_{m_1}^+ + S_{m_2}^+) \right]. \tag{109}$$

Consequently

$$[\, T^-_{(m_1, m_2) \to m_1} + T^-_{(m_1, m_2) \to m_2} \,] a(m_1, m_2) \phi(m_1, m_2) = a(m_1) \phi(m_1) + a(m_2) \phi(m_2). \tag{110}$$

Define

$$F'_{(m_1, m_2)} = [\, T^-_{(m_1, m_2) \to m_1} + T^-_{(m_1, m_2) \to m_2} \,] a(m_1, m_2) \left(\frac{\partial^2 a(m_1, m_2)}{\partial m_1 \partial m_2} \right)^{-1} \frac{\partial}{\partial m_1} \frac{\partial}{\partial m_2}, \tag{111}$$

so that

$$\left(\sum_{m_1 < m_2}^{N} F_{(m_1, m_2)} \right) | \psi_2 > = (N-1) | \psi_1 >,$$

on the other hand,

$$Q^-_{1,2} = i \sum_{j<k}^{N} W'^{(2)}_{jk} (\vec{S}_j \times \vec{S}_k)^+ + \sum_{j=1}^{N} \lambda_j^{(2)} S_j^+,$$

thus

$$Q^-_{1,2} = \frac{1}{N-1} \sum_{m_1 < m_2}^{N} F'_{(m_1, m_2)}, \tag{112}$$

25

by making comparison the coefficients of $(\vec{S}_j \times \vec{S}_k)^+$ and S_j^+ of the both side of Eq. (112), it yields

$$W'_{m_1,m_2}{}^{(2)} = \frac{1}{N-1}[a(m_1) - a(m_2)]\left(\frac{\partial^2 a(m_1,m_2)}{\partial m_1 \partial m_2}\right)^{-1}\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2}, \quad (m_1 < m_2),$$

$$\lambda_j^{(2)} = \frac{1}{N-1}\sum_{m \neq j}^{N}[a(m) + a(j)]\left(\frac{\partial^2 A(m,j)}{\partial m \partial j}\right)^{-1}\frac{\partial}{\partial m}\frac{\partial}{\partial j}. \tag{113}$$

Lowering operators for $r \geq 3$ can be obtained in the same way, they are

$$W'_{jk}{}^{(r)} = \frac{1}{N-r+1}\sum_{l_1,\cdots,l_{r-2}\neq j,k}$$

$$[A(j,l_1,\cdots,l_{r-2}) - A(k,l_1,\cdots,l_{r-2})]\left(\frac{\partial^r A(j,k,l_1,\cdots,l_{r-2})}{\partial l_1 \partial l_2 \cdots \partial l_{r-2}\partial j \partial k}\right)^{-1}\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\cdots\frac{\partial}{\partial l_{r-2}}\frac{\partial}{\partial j}\frac{\partial}{\partial k}$$

$$\lambda_j^{(r)} = \frac{1}{N-r+1}\left\{\sum_{m_1,\cdots,m_{r-1}\neq j}\left[\sum_{(l_1,\cdots,l_{r-2})\in(m_1,\cdots,m_{r-1})}A(j,l_1,\cdots,l_{r-2})\right.\right.$$

$$\left.\left. + A(m_1,m_2,\cdots,m_{r-1})\right]\left(\frac{\partial^r A(j,m_1,m_2\cdots,m_{r-1})}{\partial m_1 \partial m_2 \cdots \partial m_{r-1}\partial j}\right)^{-1}\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2}\cdots\frac{\partial}{\partial m_{r-1}}\frac{\partial}{\partial j}\right\}. \tag{114}$$

Like $Q_{r,r-1}^+$, when acts on $\mid \psi_r >$, $Q_{r-1,r}^-$ can be simplified to $\mathcal{Q}_{r-1,r}^- = \sum_{j=1}^{N}\alpha_j'^{(r)}S_j^-$, with

$$\alpha_j'^{(r)} = \frac{1}{2}\sum_{k\neq j}^{N}W'_{j,k}{}^{(r)} + \lambda_j^{(r)}. \tag{115}$$

Consequently, the lowering operators $Q_{r-1,r}^-$ or $\mathcal{Q}_{r-1,r}^-$ are also found. In particular, $\mathcal{Q}_{1,0}^+ = \sum_{m=1}^{N} a(m)S_m^-$, $\mathcal{Q}_{0,1}^-$ can be simplified to a more simple form $\mathcal{Q}_{0,1}^- = \sum_{m=1}^{N} a^{-1}(m)S_m^+$ when it acts on $\mid \psi_1 >$. These two operators are mutually adjoint. However, for general $r \geq 2$, the Hermitian properties for $\mathcal{Q}_{r,r-1}^+$ and $\mathcal{Q}_{r-1,r}^-$ are not held.